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Lappo-Danilevsky Hyperlogarithm and Period of Calabi–Yau Manifold

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We discuss the period of complex structure modulus space for a degree-$(n + 1)$ Calabi–Yau hypersurface embedded in complex n-dimensional projective space $\mathbf{CP}^n$, in view of a higher logarithm expansion. The main result is that the periods have Lappo-Danilevsky-type hyperlogarithmic structure.

1. INTRODUCTION

The Picard–Fuchs equation has appeared in several areas in mathematical physics. For example, it played a central and important role in extracting nonperturbative information in the mirror symmetry (Yau, 1992) of, e.g., the Calabi–Yau quintic hypersurface embedded in complex four-dimensional projective space $\mathbf{CP}^4$ (Candelas *et al.*, 1991), and similarly in recent studies of $N = 2$ supersymmetric Yang–Mills gauge theories (Seiberg and Witten, 1994a,b; Ito and Yang, 1996; Ohta, 1996a,b). As for the former, the Picard–Fuchs equation was a fourth-order ordinary differential equation and it turned out that it was equivalent to a certain generalized hypergeometric differential equation (Candelas *et al.*, 1991). From several other examples (Font, 1993; Klemm and Theisen, 1993), we already know that the period integrals of Calabi–Yau manifolds with complex structure moduli can be expressed similarly by the generalized hypergeometric function.

On the other hand, Lappo-Danilevsky (1958) discussed the hyperlogarithms, in his terminology, of ordinary differential equations with singularities by a method of successive approximation. Applying this hyperlogarithmic expansion to Gauss's hypergeometric equation, which can be identified with

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the Picard–Fuchs equation of a complex one-dimensional torus, we can easily observe the celebrated dilogarithm in the lower order of the expansion. This fact leads us to suspect that the period of the Calabi–Yau manifold may include a certain special function.

In this paper we develop the connection of the Lappo-Danilevsky hyperlogarithm and the period of the Calabi–Yau manifold. In Section 2 we discuss the Picard–Fuchs equation. In particular, we rewrite it as Fuchsian system of first-order differential equations with regular singularities. In Section 3 we solve it iteratively under a certain initial condition. We find that the solution, i.e., the generalized hypergeometric function, can be expressed by Lappo-Danilevsky-type hyperlogarithms. It turns out that the leading behavior is governed by a classical polylogarithm. In Section 4, we discuss its monodromy. Section 5 is a brief summary.

2. FUCHSIAN SYSTEM

The Calabi–Yau manifolds $\mathcal{M}$ which we consider in this paper are families of Fermat-type hypersurfaces embedded in $\mathbf{CP}^n$ ($n > 1$). The definition of $\mathcal{M}$ is the zero locus of

$$W(x, \psi) = \frac{1}{n+1} \sum_{i=1}^{n+1} x_i^{n+1} - \psi \prod_{i=1}^{n+1} x_i \tag{2.1}$$

where the x_i are local coordinates of $\mathbf{CP}^n$ and $\psi \in \mathbf{C}$ is a complex structure moduli. The numerical factor $1/(n + 1)$ is for convenience.

The complex structure modulus space is described by period integrals (Lerche *et al.*, 1992) of globally nonvanishing holomorphic $(n - 1)$-form defined by

$$\Pi = \oint_\gamma \frac{d\mu}{W(x, \psi)} \tag{2.2}$$

where

$$d\mu = \sum_{i=1}^{n+1} (-1)^i x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1} \tag{2.3}$$

and γ is the canonical representative of $H_{n-1}(\mathcal{M})$. Here the wedge sign denotes omission.

A technique to evaluate (2.2) is available (Berglund *et al.*, 1994), but in some cases it is better to solve the Picard–Fuchs equation rather than make a direct calculation of (2.2). In particular, from several examples (Candelas *et al.*, 1991; Font, 1993; Klemm and Theisen, 1993; D'Auria and Ferrara, 1994), we are familiar with the fact that (2.2) can be expressed by the generalized hypergeometric function, whose definition is given by

$$ {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{z^k}{k!}, \qquad |z| < 1 \tag{2.4} $$

where $(*)_i = \Gamma(* + i)/\Gamma(*)$ is the Pochhammer symbol (Slater, 1966).

This geometric series can be characterized as a solution Π_0 to the following differential equation:

$$ \left[z \frac{d}{dz} \prod_{i=1}^{q} \left(z \frac{d}{dz} + \beta_i - 1 \right) - z \prod_{i=1}^{p} \left(z \frac{d}{dz} + \alpha_i \right) \right] \Pi_0 = 0 \tag{2.5} $$

where $p = q + 1$ and $p = n$. Then the solutions near $\psi = 0$ can be obtained with the following identifications:

$$ \alpha_1 = \cdots = \alpha_p = \frac{1}{p+1}, \qquad \beta_i = \frac{i+1}{p+1}, \qquad z = \psi^{n+1}, \qquad \Pi = \Pi_0 \tag{2.6} $$

On the other hand, those near $\psi = \infty$, which is called the large-radius limit, can be obtained with

$$ \beta_1 = \cdots = \beta_q = 1, \qquad \alpha_i = \frac{i}{p+1}, \qquad z = \psi^{-(n+1)}, \qquad \Pi = z^{1/(n+1)}\Pi_0 \tag{2.7} $$

Let us try to rewrite (2.5) as a system of first-order Fuchsian differential equations. For this purpose, we define

$$ \Pi_{i+1} = z \frac{d\Pi_i}{dz}, \qquad i = 0, \ldots, p-2 \tag{2.8} $$

It is easy to find that (2.5) is equivalent to the following matrix differential equation

$$ \frac{d\mathbf{\Pi}}{dz} = \left(\frac{A_1}{z} + \frac{A_2}{1-z} \right) \mathbf{\Pi} \tag{2.9} $$

where $\mathbf{\Pi} = {}^t(\Pi_0, \ldots, \Pi_{p-1})$, both A_1 and A_2 are $p \times p$ constant matrices given by

$$ A_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 0 & -t_{p-1} & \cdots & \cdots & \cdots & -t_1 \end{pmatrix} $$

$$A_2 = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \\ s_p & u_{p-1} & \cdots & \cdots & \cdots & u_1 \end{pmatrix} \tag{2.10}$$

and

$$\begin{aligned} u_k &= s_k - t_k \\ s_i &= \sum_{k_i > \cdots > k_1 = 1}^{p} \alpha_{k_1} \cdots \alpha_{k_i} \\ t_i &= \sum_{k_i > \cdots > k_1 = 1}^{p-1} (\beta_{k_1} - 1) \cdots (\beta_{k_i} - 1) \end{aligned} \tag{2.11}$$

3. HYPERLOGARITHM AND PERIOD

We can easily solve (2.5) by a standard Frobenius method, but we would like to solve it recursively in order to see the logarithmic structure of the solution. In this and subsequent discussions, we discuss the behavior only near $z = 0$ ($\psi^{n+1} = 0$), unless otherwise noted.

Since we know that the solution to (2.9) is (2.4), we can find by using the definition (2.8) that the initial conditions at $z = 0$ for this system are given by

$$\Pi_0 = 1, \qquad \Pi_i = 0, \qquad i = 1, \ldots, p - 1 \tag{3.1}$$

Integrating (2.9) with (3.1), we obtain the integral representation of (2.9)

$$\mathbf{\Pi} = \mathbf{\Pi}_0 + \int_0^z \left(\frac{A_1}{x} + \frac{A_2}{1 - x} \right) \mathbf{\Pi} \, dx \tag{3.2}$$

where $\mathbf{\Pi}_0 = {}^t(1, 0, \ldots, 0)$ is a $p \times 1$ column matrix. Do not confuse the boldface letter ($\mathbf{\Pi}_0$) with the lightface one (Π_0).

Solving (3.2) iteratively, we find that $\mathbf{\Pi}$ can be obtained as a convergent series near the origin of $\mathbf{C}^{p+q}$ (Lappo-Danilevsky, 1958) and it can be analytically continued to any region in $\mathbf{C} - \{0, 1\}$. The result is

$$\Pi_0(z) = 1 + s_p Li_p(z) + s_p^2 L_{p,p}(z) + s_p \sum_{i=1}^{p-1} u_i L_{i,p}(z) - s_p \sum_{i=1}^{p-1} t_i Li_{i+p}(z) + \cdots \tag{3.3}$$

and for $k = 1, \ldots, p-1$,

$$\Pi_{p-k}(z) = s_p Li_k(z) + s_p^2 L_{p,k}(z) + s_p \sum_{i=1}^{p-1} u_i L_{i,k}(z) - s_p \sum_{i=1}^{p-1} t_i Li_{i+k}(z) + \cdots \tag{3.4}$$

where $Li_k(z)$ is the polylogarithm (see Appendix) and

$$L_{j,k}(z) = \underbrace{\int_0^z \frac{dx_k}{x_k} \int_0^{x_k} \frac{dx_{k-1}}{x_{k-1}} \cdots \int_0^{x_1}}_{k \text{ times}} \frac{Li_j(x_1)}{1 - x_1}\, dx_1 \tag{3.5}$$

Note that the expansion coefficients, e.g., Li_k or $L_{i,k}$, are absolute convergent functions; therefore the Π_i are indeed convergent series. The expansion coefficients are often called Lappo-Danilevsky hyperlogarithms. Originally, Lappo-Danilevsky discussed various properties of hyperlogarithms in the context of systems of singular differential equations (Lappo-Danilevsky, 1958), but explicit examples of hyperlogarithm expansions for various singular differential equations such as the generalized hypergeometric system, the Appell, or the Lauricella system do not seem to have appeared.

Since we know that the fundamental period for complex structure moduli space of the Calabi–Yau manifold corresponds to the fundamental solution to the Picard–Fuchs equation (Candelas *et al.*, 1991; Font, 1993; Klemm and Theisen, 1993, D'Auria and Ferrara, 1994) up to overall numerical factor, we can immediately conclude that the leading behavior of Π in this hyperlogarithm expansion is controlled by $Li_p(z)$ and the higher order terms can be constructed from $Li_p(z)$. Note that the index p of Li_p corresponds to the dimension of the ambient space $\mathbf{CP}^p$.

From the algorithm of successive approximation, it is easy to observe that Π_0 can be written as

$$\Pi_0(z) = 1 + \sum_{i=0}^{\infty} V_i F_i\left(\begin{matrix} 1 & a_2 & \cdots & a_i \\ 0 & 0 & \cdots & 0 \end{matrix}\middle| z\right) \tag{3.6}$$

where the V_i are some polynomial of s_*, t_* and

$$F_i\left(\begin{matrix} 1 & a_2 & \cdots & a_i \\ b_1 & b_2 & \cdots & b_i \end{matrix}\middle| z\right) = \int_{b_i}^{z} \frac{dx}{x - a_i} F_{i-1}\left(\begin{matrix} a_1 & \cdots & a_{i-1} \\ b_1 & \cdots & b_{i-1} \end{matrix}\middle| x\right) \tag{3.7}$$

or equivalently,

$$F_i\left(\begin{matrix} a_1 & \cdots & a_i \\ b_1 & \cdots & b_i \end{matrix}\middle| z\right) = \underbrace{\int_{b_i}^{z} \frac{dx_i}{x_i - a_i} \cdots \int_{b_2}^{x_3} \frac{dx_2}{x_2 - a_2} \int_{b_1}^{x_2}}_{i \text{ times}} \frac{dx_1}{x_1 - a_1} \tag{3.8}$$

The a_* are 0 or 1 due to the order of iterate integration.

4. MONODROMY

Let us consider the monodromy. From the theory of ordinary differential equations, we know that there is no nontrivial monodromy around $z = 0$ ($\psi^{n+1} = 0$) for the fundamental solution (period). However, as the fundamental period Π is now expanded by logarithmic functions, the effect of $z \to e^{2\pi i} \cdot z$ cannot be ignored for higher order terms. Therefore in view of the hyperlogarithm expansion we must treat the monodromy carefully even for $z = 0$. Note that this situation is very different from the usual convergent series expansion.

First, recall Wechsung's theorem (Wechsung, 1991), which says that the hyperlogarithms transform as

$$F_i\left(\begin{matrix} 1 & a_2 & \cdots & a_i \\ 0 & 0 & \cdots & 0 \end{matrix}\middle| z\right) \to F_i\left(\begin{matrix} 1 & a_2 & \cdots & a_i \\ 0 & 0 & \cdots & 0 \end{matrix}\middle| z\right) + \Delta_k F_i \tag{4.1}$$

under the deformation of the path starting at a point z and going around a_k (and not around a_n, $n \neq k$) counterclockwise. Here,

$$\frac{\Delta_k F_i}{2\pi i} = F_{k-1}\left(\begin{matrix} 1 & a_2 & \cdots & a_{k-1} \\ 0 & 0 & \cdots & 0 \end{matrix}\middle| a_k\right) F_{i-k}\left(\begin{matrix} a_{k+1} & \cdots & a_i \\ a_k & \cdots & a_k \end{matrix}\middle| z\right) \tag{4.2}$$

However, we are in the situation that some of a_* coincide because of $a_* = 0$ or 1. In order to include the contribution from this coincidence, we should modify (4.1) as

$$F_i\left(\begin{matrix} 1 & a_2 & \cdots & a_i \\ 0 & 0 & \cdots & 0 \end{matrix}\middle| z\right) \to F_i\left(\begin{matrix} 1 & a_2 & \cdots & a_i \\ 0 & 0 & \cdots & 0 \end{matrix}\middle| z\right) + \sum_k \Delta_k F_i \tag{4.3}$$

where the summation is over all coincident singular points.

Consequently, it follows that

$$\Pi \to \Pi + \sum_{i=0}^{\infty} \sum_k V_i \Delta_k F_i \tag{4.4}$$

However, since the fundamental period must be invariant under the monodromy around $z = 0$, as mentioned above, the second term of (4.4) must be

$$\sum_{i=0}^{\infty} \sum_{k} V_i \Delta_k F_i = 0 \tag{4.5}$$

Similar relations hold for other singularities.

5. SUMMARY

In this paper, we treated a degree-$(n + 1)$ Calabi–Yau Fermat-type hypersurface $\mathcal{M}$ embedded in $\mathbf{CP}^n$ with one complex structure modulus. We showed that the period of complex structure modulus space can be expressed by Lappo-Danilevsky-type hyperlogarithms. Using this method, we showed that the polylogarithm Li_n originates from the period of $\mathcal{M}$.

Though we treated only simple cases, i.e., one complex structure modulus models, it will be easy to extend the analysis to several complex modulus models and so on. The description by hyperlogarithms may not give any new aspects of the underlying physics, but it will be helpful for understanding the mathematical background of the modulus space of the Calabi–Yau manifold.

APPENDIX. POLYLOGARITHM

In this appendix, we briefly review the polylogarithm for the nonspecialist. The reader interested in more details should refer to Lewin (1958, 1991).

In the late seventeenth century, Leibnitz defined the dilogarithm by

$$Li_2(z) = \frac{z}{1^2} + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \cdots, \qquad |z| \leqq 1 \tag{A.1}$$

This series has an integral representation

$$\begin{aligned} Li_2(z) &= -\int_0^z \frac{\ln(1 - x)}{x}\, dx \\ &= \int_0^z \frac{dt}{t} \left(\int_0^t \frac{dx}{1 - x} \right) \end{aligned} \tag{A.2}$$

The right-hand side of (A.2) suggests that $Li_2(z)$ can be analytically continued to any region of $\mathbf{C} - \{0, 1\}$. Moreover, since both (indefinite) integrals of $1/z$ and $1/(1 - z)$ give a logarithm, $Li_2(z)$ is called the "dilogarithm."

We can generalize $Li_2(z)$ in a natural way. For example, the function $Li_n(z)$ defined by

$$Li_n(z) = \frac{z}{1^n} + \frac{z^2}{2^n} + \frac{z^3}{3^n} + \cdots, \qquad |z| \leqq 1 \tag{A.3}$$

is called the polylogarithm. When $n = 1$, $Li_1(z)$ coincides with $-\ln(1 - z)$. In the case of $n = 3$, it is called the trilogarithm.

$Li_n(z)$ can be recursively defined by

$$Li_n(z) = \int_0^z \frac{Li_{n-1}(x)}{x}\,dx \tag{A.4}$$

For the monodromy of the polylogarithm, see Ramakrishnan (1982).

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